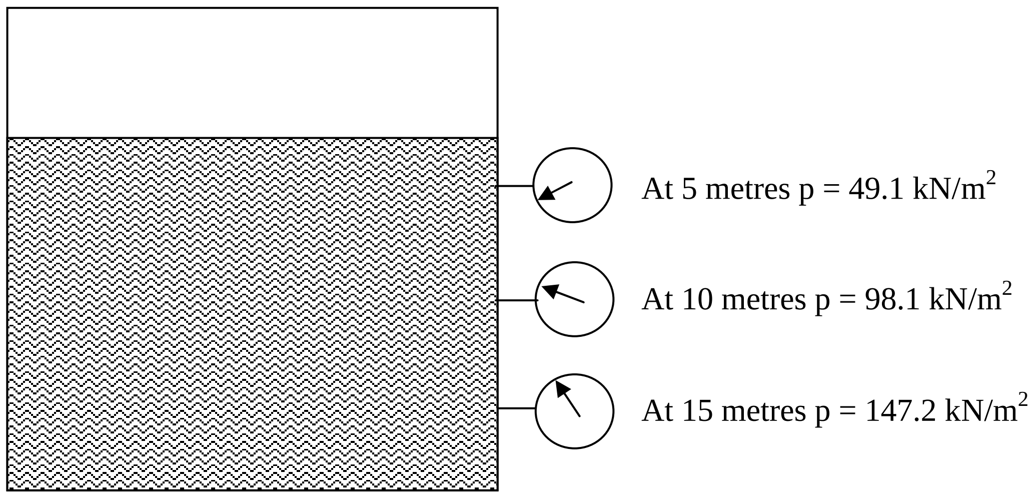
HYDROSTATICS

The study of hydrostatics is the study of fluids that have no motion. However even in the static state they can produce large forces on the surfaces that contain them.

Consider a container filled with a liquid of density ρ (kg/m^3) to a depth of h metres. If we measure the pressure at various points down the container, we will find that the pressure increase uniformally with depth. This pressure is called the static pressure and equates to ρgh, where h is the depth of the liquid to the point of the pressure measurement.

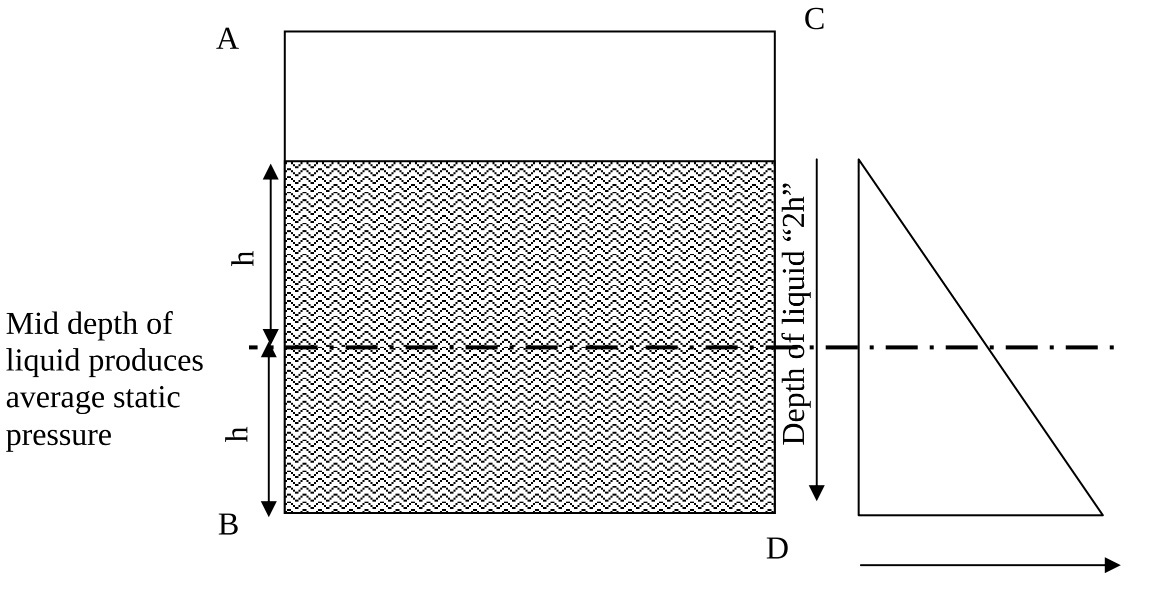
In this example, if the liquid is fresh water of density 1000 kg/m^3, then the pressure at 5, 10 and 15 metres water depth will be as shown.



The pressure on the liquid is assumed to act normal to the surface, and produces a uniform pressure at the same depth.

What do you think will happen if the air pressure above the liquid was now increased? This will have the effect of pressing the liquid down, and will increase each pressure gauge by the same amount as the rise in pressure. So if the air pressure was 45 kN/m^2 or 0.45 bar, then the new gauge readings would be 94.1, 143.1, and 192.2kN/m^2 respectively.

Once we know the pressure at a certain depth of liquid, we can now find the force that that liquid will exert of the sides of the container that holds it. However we have already seen that the pressure changes due to depth, so to find the force we must first find the average pressure. This will occur at *mid-depth*, so we can see by using the triangle to represent the various pressures on the side of the container.



From Force F = Pressure (p) x Area (A), then the total force on the side ABCD will be the product of the area of ABCD and the average pressure exerted on that side by the liquid.

Height of liquid to produce the average pressure = $\frac{2h}{2} = h$

So **Force = ρghA**

Example

If the fresh water carried in a tank of length of side 4m is 3m deep, calculate the force exerted on the side of the tank.
Take the density of fresh water as 1000kg/m^3 at all times unless otherwise informed.

Mid depth of liquid is $\frac{3m}{2}$ = 1.5m

Area of tank side = 4 x depth of water = 4 x 3 = 12m^2

Note this area is the same as the "wetted area"

Force = ρghA = 1000 x g x 1.5 x 12 = 176.58kN

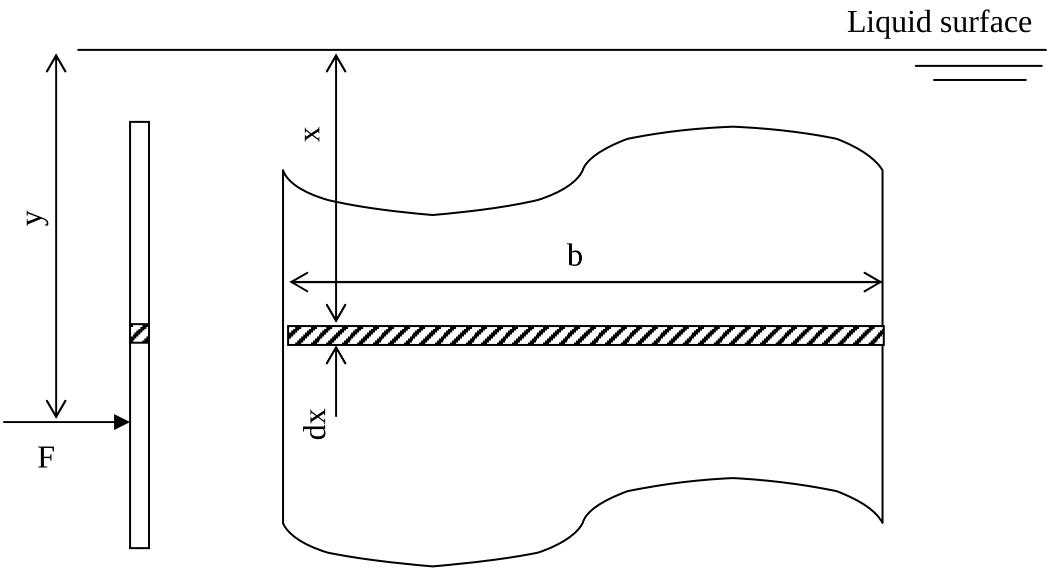
SAQ #1

For a tank of equal side length 3m x 3m filled to a level of 1.4m with fresh water.
Calculate the force exerted on the side of the tank.
ANS at the end of the Hydrostatics unit.

Centre of pressure

Although we have calculated the force exerted by the liquid pressure, we also need to know the point of action of this force. This point is classified as the Centre of Pressure, and is defined as the point where the resultant force can be assumed to act.

Although we have taken the mid depth for the average pressure calculations, we need to equate the various pressure moments about the free surface to locate this centre of pressure location (COP)



Consider a plate that is immersed in a liquid of density ρ. The force F acts at the centre of pressure "y" from the surface. To find the dimension "y" we will equate the sum of the moments of all thin strips to that of the product Fy about the liquid surface.

For a single strip of depth dx and width b

Force = pressure x area of strip

= p (b dx)

= (ρgx) (b dx)

= ρ.g.x.b.dx

The moment of this force about the surface = force x moment

= Fx

= (ρ.g.x.b.dx) x

= $\rho.g.x^2.b.dx$

The total moment of all strips about the surface will be the integral of this single moment = $\int \rho.g.x^2.b.dx$

This total moment is to be equated to Fy, so Fy = $\int \rho.g.x^2.b.dx$

However the term $\int x^2.b.dx$ is the second moment of area about the free or liquid surface (I_{oo}) which from the parallel axes theorum is defined as $I_{GG} + Ah^2$

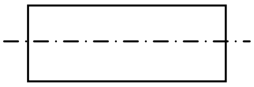
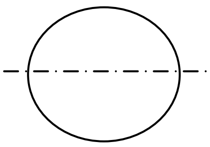
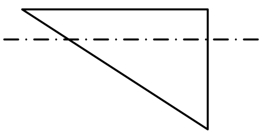
Thus $Fy = \rho.g.I_{oo}$

or substituting with $F = \rho ghA$

Then $\rho ghAy = \rho gI_{oo}$

So $y = \frac{I_{oo}}{Ah}$ or $y = \frac{I_{GG} + Ah^2}{Ah}$ or $y = \frac{I_{GG}}{Ah} + h$

The value of I_{GG} for standard shapes are listed below and should be memorised.

Shape	Centre of area from the **top** of the area (h)	I_{GG}
Rectangle 	$\frac{d}{2}$	$\frac{bd^3}{12}$
Circle 	$\frac{d}{2}$	$\frac{\pi d^4}{64}$
Triangle 	$\frac{d}{3}$	$\frac{bd^3}{36}$

Example

For a rectangular tank of dimensions 3m wide by 2.4m depth, calculate the centre of pressure from the top edge, when the tank is full.

$h = \frac{d}{2} = \frac{2.4}{2} = 1.2m$

$I_{GG} = \frac{bd^3}{12} = \frac{3 \times 2.4^3}{12} = 3.456m^4$

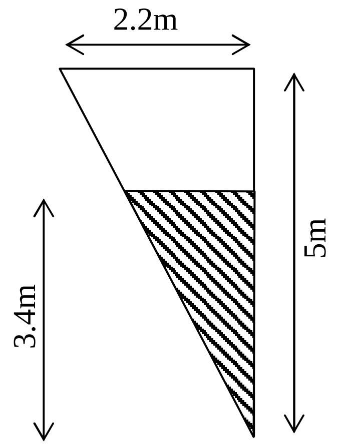
Area $= bd = 3 \times 2.4 = 7.2m^2$

So $y = \frac{I_{GG}}{Ah} + h = \frac{3.456}{7.2 \times 1.2} + 1.2 = 0.4 + 1.2 = 1.6m$

So for a **rectangular plate that is just immersed**, the centre of pressure will be **2/3rd of the liquid depth**.

<u>Example</u>

Consider a triangular side of an aft peak tank as shown in Fig 3. The tank is filled to a depth of 3.4m with sea water of density 1025 kg/m^3. Calculate the force on the tank side, and the centre of pressure from the water level.



STEP ONE - Obtain the wetted area of tank
Area of triangle = depth of liquid x maximum width/2
Maximum width of the water depth can be found from similar triangles as the "shape" of the water and the shape of the tank is similar.

So maximum width of water = $\frac{3.4 \times 2.2}{5}$ = 1.496m

Hence area of the triangle will be $\frac{3.4 \times 1.496}{2}$ = 2.543m^2

STEP TWO – Obtain the h value and the static force
Force = ρghA

h is the centre of area of the water depth, which for a triangle is $\frac{d}{3}$ (see page 4).

So h = $\frac{3.4}{3}$ = 1.133m (remember this is measured from the surface)
Thus F = 1025 x g x 1.133 x 2.543 = 28.97kN

STEP THREE – Obtain the values of second moment of area, and then COP
This force will act at the centre of pressure, and the COP can be found in stages

I_{GG} for a triangle is $\frac{b \times d^3}{36} = \frac{1.496 \times 3.4^3}{36}$ = 1.633m^4

We have already calculated the centre of area depth and area, so the calculation of COP will be

$$y = \frac{I_{GG}}{Ah} + h \quad = \quad \frac{1.633}{2.543 \times 1.133} + 1.133 \quad = \quad 1.7\text{m}$$

So for a **triangular plate that is just immersed**, the centre of pressure will be ***half of the liquid depth.***

So far we have been dealing with the forces and COP of various shapes when the whole shape is just immersed. However in many practical situations the area under consideration is immersed well below the surface of the liquid. A similar procedure is carried out, BUT we must take care when calculating the depth from the surface to the centre of area.

Example

A circular manhole door is fitted in a ballast tank filled with sea water. The door is 1.6m in diameter, and the top of the door is 5m below the level of the water. Find the force on the door from the hydrostatic pressure and the COP from the level of the water.

STEP ONE - Obtain the wetted area of the door

Area of circle $= \dfrac{\pi d^2}{4} = \dfrac{\pi \times 1.6^2}{4} = 2\text{m}^2$

STEP TWO – Obtain the h value and the static force

h is the distance from the water level to the centre of area. For a circle the centre of area is in the centre of the circle or $\dfrac{d}{2}$, so h = $5 + \dfrac{1.6}{2}$ = 5.8m

Force = ρghA

Thus F = 1025 x g x 5.8 x 2 = 116.64kN

STEP THREE – Obtain the values of second moment of area, and then COP

This force will act at the centre of pressure, and the COP can be found in stages

I_{GG} for a circle is $\dfrac{\pi d^4}{64} = \dfrac{\pi 1.6^4}{64}$ = 0.322m^4

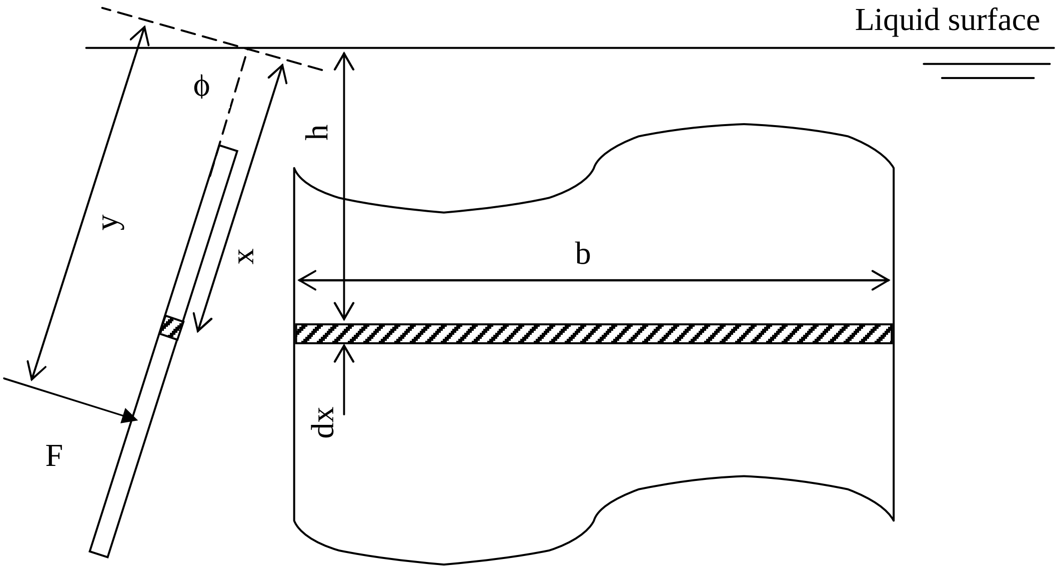
We have already calculated the centre of area depth and area, so the calculation of COP will be

$$y = \frac{I_{GG}}{A \times h} + h \quad = \quad \frac{0.322}{2 \times 5.8} + 5.8 \quad = \quad 5.828\text{m}$$

You should observe that the deeper the area is below the liquid surface, then the COP moves closer to the centre of area. When the **circular plate is just immersed**, then the **COP will be 5/8th of the liquid depth**.

Inclined plates

We must also consider the forces that act on inclined plates, such as hopper sides on upper ballast tanks.



Consider a plate that is immersed in a liquid of density ρ, and inclined at an angle φ to the surface. The force F acts normal to the plane at y measured along the plane from the surface. To find the dimension y we will equate the sum of the moments of all thin strips to that of the product Fy about the liquid surface.

For a single strip of depth dx and width b

Force = pressure x area of strip
= p (b dx)
= (ρgh) (b dx)
= ρ.g.h.b.dx (where the h dimension is the vertical depth from the surface)

The moment of this force about the surface = force x moment
= F x
= (ρ.g.h.b.dx) x

As x.sin φ = h, then F = $\rho.g.x^2.b.\sin\phi.dx$

The total moment of all strips about the surface will be the integral of this single moment = $\int \rho.g.x^2.b.\sin\phi.dx$

This total moment is to be equated to Fy, so Fy = $\int \rho.g.x^2.b.\sin\phi.dx$

However the term $\int x^2.b.dx$ is the second moment of area (I_{oo}) about the free or liquid surface, which from the parallel axes theorem is defined as $I_{GG} + Ah^2$

Thus Fy = $\rho.g.\sin\phi.I_{oo}$

or substituting with F = ρghA = ρgA.x.sinϕ

Then ρgA.xsinϕ.y = ρg.sinϕ.I_{oo}

So $y = \frac{I_{oo}}{Ax}$ or $y = \frac{I_{GG} + Ax^2}{Ax}$ or $y = \frac{I_{GG}}{Ax} + x$

where x is the distance to the *centre* of the wetted area

These are the same equations as we obtained for the vertical planes BUT this time we have measured x and y *along the plane*.
Only the h value is measured vertically.

Example

A square door of equal sides 1.4m is fitted on a sloping tank plate with its top edge 2m vertically below the level of fresh water. The tank plate is declined at 60^o to the horizontal.
Calculate the force on the door, and the centre of pressure measured both along the plane and vertically from the water level.

STEP ONE - Obtain the wetted area of the door
Area of door = 1.4^2 = $1.96m^2$

STEP TWO – Obtain the h value and the static force
h is the vertical distance from the water level to the centre of area. For a square the centre of area is in the centre of the square or $\frac{d}{2}$, so h = 2 + $\frac{1.4 \sin 60^o}{2}$ = 2.606m

Force = ρghA
Thus F = 1000 x g x 2.606 x 1.96 = 50.11kN

STEP THREE – Obtain the values of second moment of area, and then COP
This force will act at the centre of pressure, and the COP can be found in stages

I_{GG} for a square is $\frac{l^4}{12} = \frac{1.4^4}{12}$ = $0.32m^4$

x **measured along the plane** = $\frac{h}{\sin \phi} = \frac{2.606}{\sin 60^o}$ = 3.01m

We have already calculated the centre of area depth and area, so the calculation of COP will be

y = $\frac{I_{GG}}{A \times x} + x$ = $\frac{0.32}{1.96 \times 3.01} + 3.01$ = 3.064m

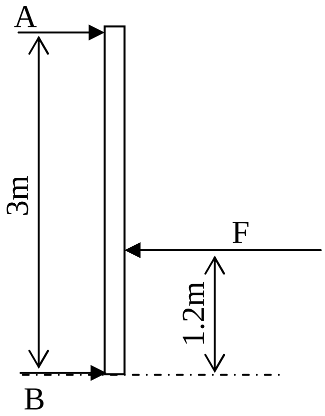
Note that y is also measured along the plane, so the vertical depth of the COP will be ysinϕ = 3.064 x sin60^o = 2.654m

SAQ #2
The end of a fresh water tank is declined at 60^o to the horizontal, and contains a square door of sides 2m long. The water level is 3.4m above the lower edge of the door.
Calculate the force and the point where this force acts (COP).
ANS at the end of the Hydrostatics unit.

Forces on tank doors & lock gates

The final element in the evaluation of the hydrostatic force on the restraining material, is the calculation of the forces on the various fasteners fitted, such as doors, manhole covers, etc.

Consider the forces acting on a single plate as shown.



If the force F on the door is found to be 58kN, what is the forces on the two restraining bolts A and B?

In this situation **we shall evaluate the forces and the moments on the door.**

This process is fundamental when trying to find the unknown forces

From the evaluation of the forces, we can see that the force F must equal A + B, as the plate is not moving in the horizontal direction.

From the evaluation of the moments, we shall equate the moments about the point B, so that A x 3 = F x 1.2, as the plate is not rotating.

So as F = 58kN, then from the moment equation A = $\frac{1.2 \times 58}{3}$ = 23.2kN

Equating the moments about A, gives F x 1.8 = B x 3, so B = $\frac{58 \times 1.8}{3}$ = 34.8kN

We can now check the force equation, and by adding A + B we should get the force quantity of 58kN. Thus 23.2 + 34.8 = 58kN

Note that the forces A and B are the only restraining forces holding the door. If the door was supported at both sides by a hinge and two bolts, then the hinge and the bolts would carry equal loads, as we will see in the next example.

Example

Consider a gate fitted to a fresh water canal lock. The gate seals a 6m wide canal, which is filled to a depth of 3.6m. The gate is hinged at 1.0m and 4.6m from the canal floor.
Calculate the force on the lock gate, and the force on each hinge.

STEP ONE - Obtain the wetted area of the lock gate
Area of gate = 6 x 3.6 = 21.6m^2

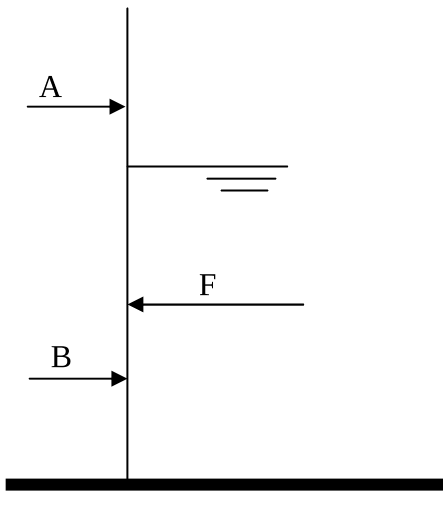
STEP TWO – Obtain the h value and the static force
h is the vertical distance from the water level to the centre of area.
For a rectangular gate the centre of area is in the centre of the gate or $\frac{d}{2}$,

so h = $\frac{3.6}{2}$ = 1.8m
Force = ρghA
Thus F = 1000 x g x 1.8 x 21.6 = 381.4kN. This will be the combined force on both the two hinges, AND the opposite side of the lock gate.

STEP THREE – Obtain the values of second moment of area, and then COP
This force will act at the centre of pressure, and the COP can be found in stages
I_{GG} for a rectangle is $\frac{bd^3}{12} = \frac{6\times 3.6^3}{12}$ = 23.328m^4

We have already calculated the centre of area depth and area, so the calculation of COP will be
y = $\frac{I_{GG}}{A\times h}+h$ = $\frac{23.328}{21.6\times 1.8}+1.8$ = 2.4m measured from the water level, or 1.2 m from the canal floor.

If you can recall we stated earlier in this section that for a rectangular plate that is just immersed, the centre of pressure will be 2/3rd of the liquid depth. So in this example the COP will be 2/3rd x 3.6m = 2.4m



Taking moments about the hinge B

(A x 3.6) – (F x 0.2) = 0, as F = 381.4kN
Thus A = $\frac{21.2}{2} = 10.6$ kN
Taking moments about the hinge A

(F x 3.4) – (B x 3.6) = 0
Thus B = $\frac{360.2}{2} = 180.1$kN
Check A+B = 10.6 + 180.1 = 190.7 kN = F

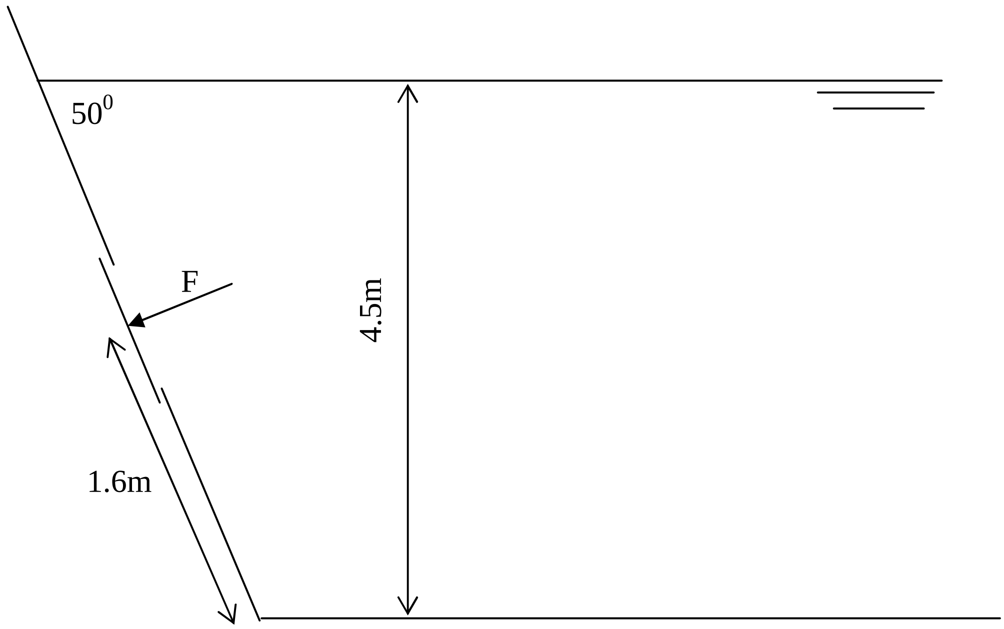
DID YOU REMEMBER THAT THE HINGES CARRIES HALF THE TOTAL FORCE WHEN THE GATE IS SHUT ?

Example – a typical examination question

A 800mm square access door is fitted to the inclined side of a domestic fresh water storage tank, as shown. The centre of the door is 1.6 m above the tank floor, as measured up the inclined side.

The door is secured by four bolts, two fitted at the top, and two fitted at the bottom edge. To achieve a positive seal the bolts compress a rubber joint by each applying a load of 500N. If the sounding in the tank is 4.5 m, calculate

a) The force on the door
b) The position of the centre of pressure
c) The load on both the top and bottom bolts



STEP ONE - Obtain the wetted area of the door

Area of door = 0.8^2 = $0.64m^2$

STEP TWO – Obtain the h value and the static force

h is the vertical distance from the water level to the centre of area.

The vertical distance from the floor to the door centre is 1.6 sin 50^0 = 1.226m, so h = 4.5 – 1.226 = 3.274m

Force = ρghA

Thus F = 1000 x g x 3.274 x 0.64 = 20.56kN

STEP THREE – Obtain the values of second moment of area, and then COP

This force will act at the centre of pressure, and the COP can be found in stages

I_{GG} for a square is $\frac{length^4}{12} = \frac{0.8^4}{12}$ $= 0.034m^4$

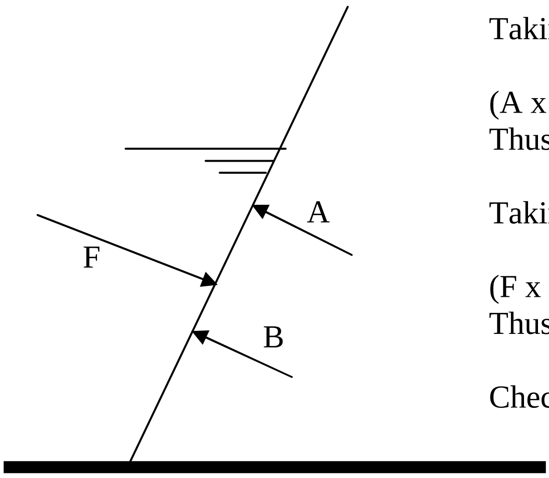
x **measured along the plane** = $\frac{4.5}{\sin\phi} - 1.6 = \frac{4.5}{\sin 50^o} - 1.6 = 4.274m$

We have already calculated the centre of area depth and area, so the calculation of COP will be

$$y = \frac{I_{GG}}{A \times x} + x \quad = \quad \frac{0.034}{0.64 \times 4.274} + 4.274 \quad = \quad 4.286\text{m}$$

Although in this case the y value and x value are very similar we can not ignore this difference as we are trying to show that we understand the method used to obtain these values.

In this example, we have two bolts at position A, and the other two bolts at position B.
The dimension from A to B is 0.8m (the height of the door), and the dimension from F to B is 388mm, and F to A is 412mm, as we have already determined that y is 4.286m along the plane, and x is 4.274m along the plane, a difference of 12mm.



Taking moments about B

(A x 0.8) – (F x 0.388) = 0, as F = 20.56kN
Thus A = 9.97kN

Taking moments about A

(F x 0.412) – (B x 0.8) = 0
Thus B = 10.59kN

Check A+B = 9.97 + 10.59 = 20.56 = F

Hence for each bolt at A there will be 50% of the total tension PLUS the joint compression force of 500N, hence each bolt at A will be loaded in tension to 9.97/2 + 0.5 = 5.49kN

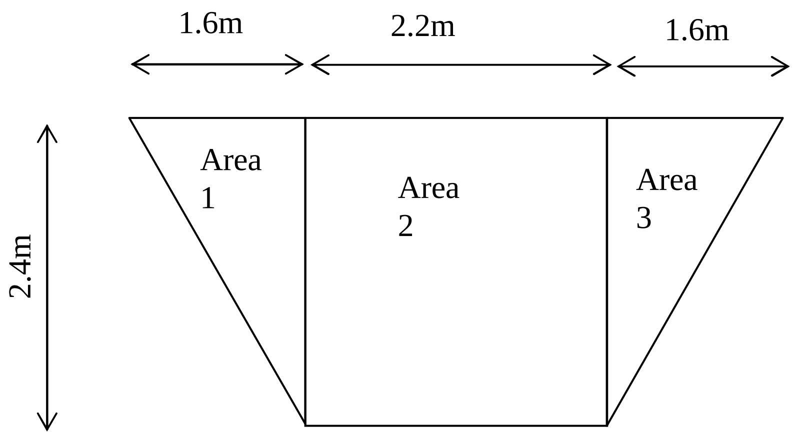
Also for each bolt at B there will be 50% of the total tension PLUS the joint force, hence bolts at B are loaded in tension to 10.59/2 + 0.5 = 5.8kN

SAQ #3
A tank of dimensions 6 m long, 3 m wide, and 5 metre deep is filled with fresh water. A pressure test is applied by filling the tank into the air vent pipe to give an additional head of 800mm. A 500mm circular access door is fitted with its centre 600mm from the tank floor. The access door is hinged at its top edge and secured by a single bolt at its lowest edge.
Calculate
a) The force on the access door
b) The tension on the bolt
b) The maximum permissible depth of water in the tank, if the bolt tension is limited to 4kN

ANS at the end of the Hydrostatics unit

The final part of this hydrostatic section is to find the force and COP when the restraining material is not in either of the standard shapes of a triangle, square, or circle.

To find the total force and COP, we will find the force and COP for each section. For the tank shape shown, the tank is filled with sea water of ρ = 1025kg/m^3.



	Area	h value	Force	I_{GG}	Ah	y
Area 1	1.92	0.8	15.44	0.6144	1.536	1.2
Area 2	5.28	1.2	63.71	2.534	6.336	1.6
Area 3	1.92	0.8	15.44	0.6144	1.536	1.2
TOTAL			Σ 94.59kN			

Taking moments about the water line to find the effective COP for the total force of 94.59kN.

(94.59 x COP) = (15.44 x 1.2) + (63.71 x 1.6) + (15.44 x 1.2)

$$\text{So COP} = \frac{18.528+101.936+18.528}{94.59} \quad = \quad 1.47\text{m}$$

Student self-test

Complete these questions, and check your answers with those given.

SAQ

A bulkhead closing one end of a floating dock is 9 m wide at the bottom and 18 m wide at the top and is 9 m deep. If submerged up to its upper edge, what is the total thrust on the bulkhead and what will be the depth to the centre of pressure. ρ for sea water = 1024 kg/m^3.

ANS 4.9 kN, 5.62 m

SAQ

A vertical bulkhead divides a tank 9 m wide. On one side of the bulkhead there is a fresh water to a height of 6.3 m and on the other side there is oil of d = 0.85 to a height of 4.5 m. Find the resultant thrust on the bulkhead and the position of the centre of pressure above the bottom of the tank.

ANS 990 kN, 2.56 m

SAQ

A tank is divided into two compartments by a vertical bulkhead. One compartment contains fresh water to a depth of 1.83 m, whilst the other holds oil to a depth of 1.37 m. The relative density of the oil is 0.958 and the width of the tank is 6.1 m. Determine the resultant thrust on the bulkhead, and the position of the centre of pressure above base of the tank.

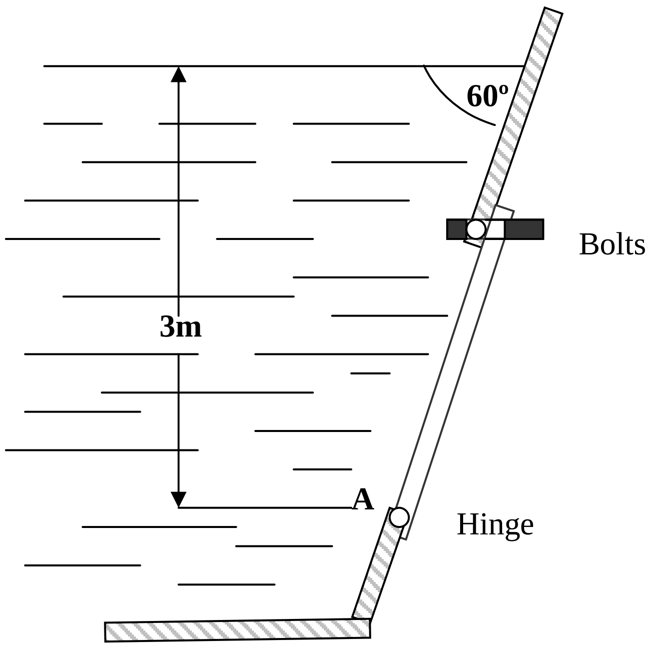
ANS 46.4 kN, 0.79 m

SAQ

The end of a tank is inclined at 60^o to the horizontal and within it is an opening 2 m square.

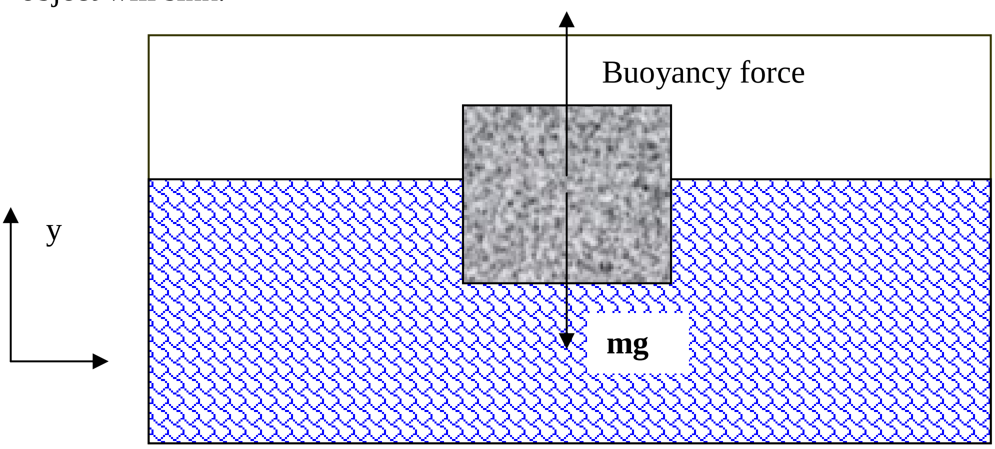
The opening is closed by a door as shown. The door has a hinge at its lower edge and is bolted at its upper edge. The tank contains water to a depth of 3m above the lower edge of the opening calculate the force applied by the bolts to overcome the fluid thrust.

ANS 36.2 kN



Archimedes's Principle

When a solid body is floating in a liquid, then the immersion of the body will displace the liquid. The volume of liquid displaced will produce a buoyancy force that is equal and opposite to the gravitational force of the body when the body is freely floating. If the gravitational force is greater than the buoyancy force then the object will sink.



Consider a body of mass 10 kg that is floating in fresh water. We already know that the gravitational force is mg or 10g in this case, which equates to 98.1N.

The **buoyancy force = ρVg**

Where ρ the density of the liquid and V is the volume displaced

Consider the forces in the y direction

$\rho Vg - mg = 0$

$1000Vg - 10g = 0$

Thus the 10kg mass will displace 10/1000 or $0.01m^3$

Example

Consider a cube of equal sides of 400mm of mass 25kg. When this is freely floating in a liquid of density $1025kg/m^3$, calculate the % of cube exposed above the water level.

From $\rho Vg - mg = 0$, then $1025Vg - 25g = 0$

So $V = \frac{25}{1025} = 0.0244m^3$

If the cube was fully immersed, then the displaced volume would be 0.4^3 or $0.064m^3$, but as it is partly immersed the volume displaced will be 0.4^2 x the submerged depth "d".

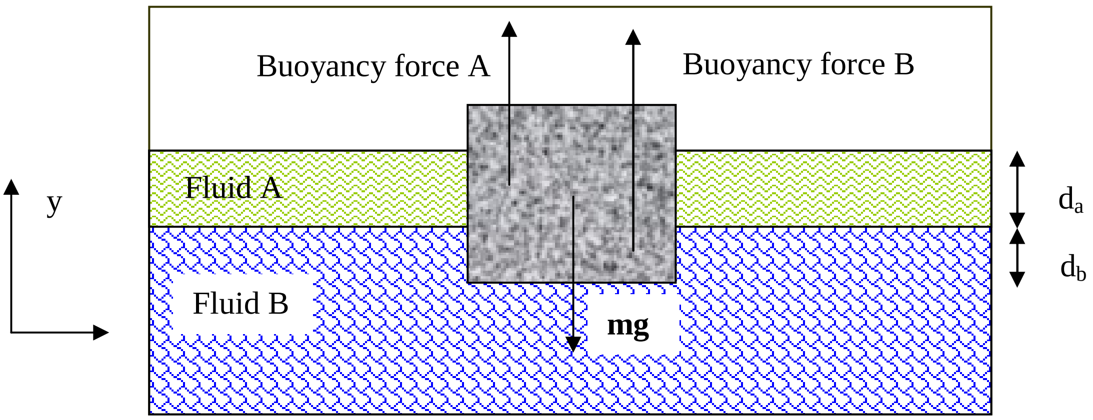
So for this example $V = 0.0244 = 0.4^2 \times d$

Thus d = 0.1525. Hence exposed cube length will be 0.4 – 0.1525 = 0.2475m.

Thus the percentage of the cube exposed will be $\frac{0.2475}{0.4} = 61.88\%$

Immiscible fluids

Normally we would assume that the liquid in which a body floats is a uniform fluid of constant density, it is possible for a lighter liquid to float on top of another liquid. If a body now floats in this mixture, then although the body's gravitational force would not alter, the buoyancy force would be the resultant of the buoyancy force of both fluids.



Example

Consider a cube of equal sides of 400mm of mass 25kg. This is freely floating in an immiscible mix of liquids of density 1000kg/m^2 on top of a liquid of density 1025kg/m^3. If the layer of the fresh water is 120mm deep, what is the percentage of cube exposed above the water level.

We would expect that for this similar cube the cube would sink further as there is a layer of lighter fluid in which it is floating.

Analyse the forces in the y direction $\rho_a V_a g + \rho_b V_b g - mg = 0$

The buoyancy force for the fresh water = $\rho_a V_a g$, but the volume of the fresh water is known as the area of the cube x depth of fresh water or $0.4^2 \times 0.120 = 0.0192\text{m}^3$

The buoyancy force for the sea water = $\rho_b V_b g$, but the volume is unknown and equals area of the cube x immersed depth in the sea water or $0.4^2 \times d_b = 0.16d_b$

From the force analysis

$(1000 \times 0.0192 \times g) + (1025 \times 0.16d_b \times g) – 25g = 0$

$188.35 + 1608.8d_b – 245.25 = 0$

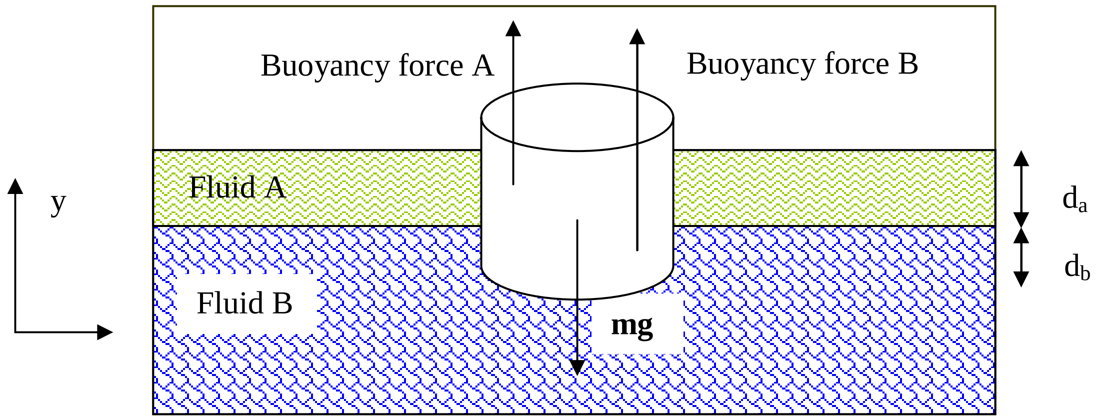
So $d_b = 0.0354m$
Thus total depth of immersion
= depth of liquid A + depth of liquid B
= 0.0354 + 0.12 = 0.155m

Height of cube exposed = 0.4 – 0.155 = 0.245m

Thus the percentage of the cube exposed will be $\frac{0.245}{0.4}$ = 61.16%

As expected the cube has sunk slightly due to the lighter liquid in which the cube is floating.

<u>Example</u>
A cylinder with flat ends floats upright in an immiscible mixture of two liquids. The upper layer is light oil of density 880kg/m^3, whilst the lower liquid is sea water of density 1025kg/m^3. The cylinder mass is 220kg, and has a diameter of 450mm, and depth 1.5m. If the cylinder floats with only 100mm exposed, calculate the depth of the lighter layer.
Calculate the mass of steel of density 7900 kg/m^3 required to be fitted beneath the cylinder to cause the cylinder to float with the top just submersed.



Equate the forces in the y direction
$\rho_a V_a g + \rho_b V_b g - mg = 0$

V_a = cylinder area x d_a $= \frac{\pi d^2}{4} \times d_a = \frac{\pi \times 0.45^2}{4} \times d_a = 0.159 d_a$

V_b = cylinder area x d_b $= \frac{\pi d^2}{4} \times d_b = \frac{\pi \times 0.45^2}{4} \times d_b = 0.159 d_b$

But we can equate the depth of the sea water to be the difference between the cylinder depth and the combined depth of the oil and exposed cylinder.

So $d_a + d_b$ + exposed length (0.1m) = total cylinder length (1.5m)

So $d_b = 1.4 – d_a$, hence $V_b = 0.159(1.4 – d_a)$ or $0.223 – 0.159d_a$

Equating the forces gives
$\rho_a V_a g + \rho_b V_b g - mg = 0$

$(880 \times 0.159d_a \times g) + (1025 \times (0.223 – 0.159d_b) \times g) – 220g = 0$

$1372.6d_a + 2242.3 – 1598.8d_a – 2158.2 = 0$

$(2242.3 – 2158.2) = d_a(1598.8 – 1372.6)$

So $d_a = 0.372$ m

When the additional steel mass is added, then will produce both a gravitational force AND a buoyancy force due to its displaced liquid (sea water in this case). As the cylinder is now submerged the depth of the sea water displaced will be
1.5 – 0.372 = 1.128 m.

So equating the forces gives
$\rho_a V_a g + \rho_b V_b g + \rho_b V_s g – m_{cyl} g – m_s g = 0$

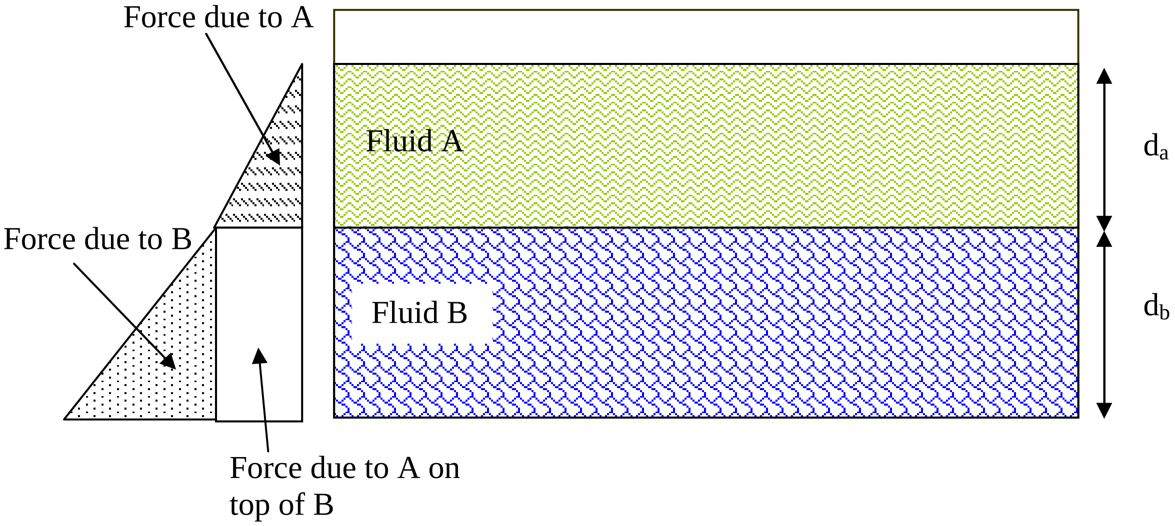
$(880 \times 0.159 \times 0.372 \times g) + (1025 \times 0.159 \times 1.128 \times g) + (1025 \times \frac{m}{7900} \times g) – 220g – mg = 0$

$510.6 + 1803.43 + 1.27m – 2158.2 – 9.81m = 0$

$155.84 = m(9.81 – 1.27)$

So m = 18.25 kg

Effects of immiscible fluids on static head



Consider two fluids contained within a tank of side length “z” metres. If we consider the fluid A to be oil of density 860 kg/m^2, then the force F_A exerted on the tank side will be

$$F_A = \rho g h A$$
$$= 860 \times g \times \frac{d_a}{2} \times (d_a \times z)$$

This force is the same as we would have derived earlier. However when we calculate the force exerted by the fluid B, then we must also consider the effects on fluid A on top of fluid B, which will increase the force exerted by fluid B.

One method is to consider the fluid A and B as a single fluid of density ρ_b. Hence we must convert the applied head of fluid A in terms of fluid B. As fluid A will always be lighter than fluid B (it is floating on top of it!), then the equivalent head of fluid A in terms of fluid B will be $d_a \frac{\rho_a}{\rho_b}$. This will produce a smaller numerical value of head than the numerical sum of $d_a + d_b$.

The force of fluid B (fresh water) will be

$$F_B = \rho g h A$$
$$= 1000 \times g \times (d_a \frac{\rho_a}{\rho_b} + \frac{d_b}{2}) \times (d_b \times z)$$

Example

Calculate the total force when the oil depth is 1.4m, and the fresh water depth is 3.2m, when the tank has a side dimension of 4m.

Force from A $= \rho g h A$

$= 860 \times g \times \frac{1.4}{2} \times (1.4 \times 4)$

$= 33kN$

Equivalent depth of oil in terms of fresh water $= 1.4 \times \frac{860}{1000}$

$= 1.204m$

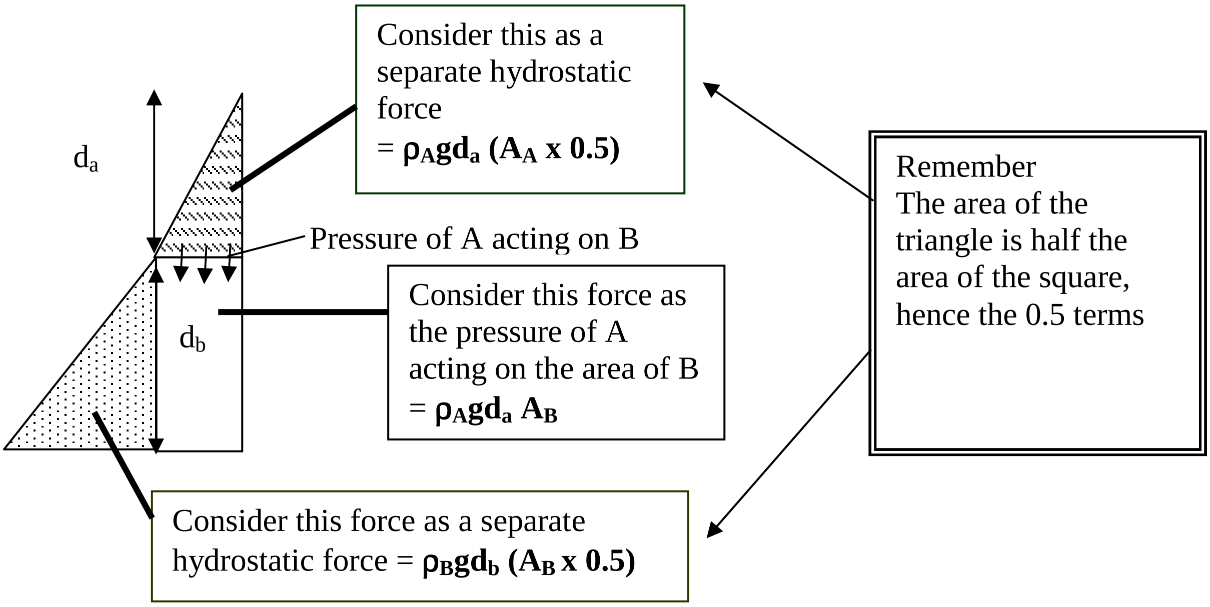
Force from B $= \rho g h A$

$= 1000 \times g \times (1.204 + \frac{3.2}{2}) \times (3.2 \times 4)$

$= 352.1kN$

Total force $= 352.1 + 33 = 385.1kN$

An alternate method is to consider the force exerted on the tank side in terms of the force triangle as shown.



The pressure at the interface of the liquids is $\rho g d_a$ or 860g1.4 = 11.81kN/m^2

Hence the force exerted by A
= pressure $_A$ x (area $_A$ x 0.5)
= 11810 x (4 x $\frac{1.4}{2}$)
= 33kN

The force exerted by A on top of B
= pressure $_A$ x area $_B$ = $\rho_A g d_a A_B$ =
= 11810 x (4 x 3.2)
= 151.2kN

The pressure exerted by B alone at the base of the tank = $\rho g d_b$
= 1000g3.2
= 31.4 kN/m^2

The force exerted just by B alone
= pressure $_B$ x (area $_B$ x 0.5)
= 31400 x (4 x $\frac{3.2}{2}$) = 200.9kN

Total force = 200.9 + 33 + 151.2 = 385.1 kN, which is the same as that obtained using the equivalent head method.

The centre of pressure can be found by equating the individual forces about the tank base or surface from the standard relationship of

$$COP = \frac{\sum F_A COP_A + F_B COP_B + +}{F_A + F_B + +}$$

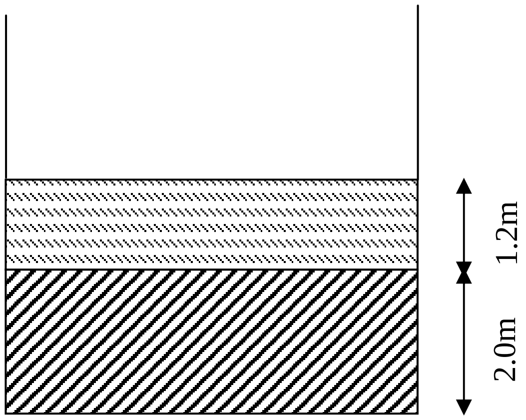
Example – a typical examination question

A tank contains fresh water to a depth of 2.0 metres. Oil of depth 1.2 metres floats on top of the fresh water. The width of the vertical tank wall is 3 metres.

Calculate the following:

a) The total hydrostatic force on the vertical tank wall
b) The position of the resultant centre of pressure
c) The equivalent depth of fresh water alone that would exert the same hydrostatic force as the two liquids combined

Density of the oil = 720 kg/m^3



The oil will exert a pressure equal to its static head or ρghA

$= 720 \text{ x g x } \frac{1.2}{2} \text{ x } (1.2 \text{ x } 3) = 15.26\text{kN}$

The water will also exert a pressure but this will be equal to its static head plus the additional head of the oil pressing down onto it. This is the same as if a gas pressure was present on top of the water.

We will equate the extra pressure of the oil head pressure of 1.2m in terms of water head, which is a heavier density.

So additional head on top of the water = $\frac{720}{1000}$ x 1.2 = 0.864m

Thus the total head on the water will be the $\frac{water \quad depth}{2}$ + additional head

$= \frac{2}{2} + 0.864 = 1.864\text{m}$

So static force = ρghA = 1000 x g x 1.864 x (2 x 3) = 109.72kN

Hence the total force = 15.26 + 109.72 = 124.98kN

We now have two forces of the oil and water acting at differing centres of pressure. We need to equate these into one centre of pressure where the total force of 124.98kN can be said to act.

<u>Using the equivalent head method</u>

The COP of the oil will be 2/3rd of the depth = 2/3 x 1.2 = 0.8m below the oil surface.

To find the COP of the water we shall use the second moment of area of the water about the oil surface.

From y = $\frac{I_{GG}}{A \times h} + h$, where h = 1.864, I_{GG} = $\frac{bd^3}{12}$ for a rectangle

then y = $\frac{3 \times 2^3}{12 \times 3 \times 2 \times 1.864} + 1.864$ = 2.043m in water head

However we must convert this distance into water and oil head, or actual measurements. The difference between these terms will be the difference used earlier where the additional head was quoted as 0.864m rather than the actual 1.2m of oil head, so the actual measurement from the oil surface will be 2.043 plus the difference of (1.2 – 0.864), so COP of the water force is 2.043 + 0.336 = 2.379m

Taking moments about the oil surface for the total force

F x COP = (F_{oil} x COP) + (F_{fw} x COP)

124.98 x COP = (15.26 x 0.8) + (109.72 x 2.379)

So COP = 2.186m

<u>Using the force triangle method</u>

Force of oil = 15.26kN

acting at 1/3rd from bottom of triangle = $\frac{1.2}{3} + 2.0$ or 2.4 m from the bottom of tank

Force of oil on water = p_{oil} x A_{water} = (720g1.2) x (2 x 3) = 50.86kN

acting at its centroid or 1.0m from the bottom of tank

Force of water = $\rho_w g d_w$ (A_w x 0.5) = 1000d 2 (2 x 3 x 0.5) = 58.86 kN

acting at 1/3rd from bottom of triangle = $\frac{2.0}{3}$ or 0.66 m from the bottom of tank

Equating the forces acting about the bottom of the tank

Total force x COP = (15.26 x 2.4) = (50.86 x 1) + (58.86 x 0.66)

124.98 x COP = 36.62 + 50.86 + 38.85

Thus COP = 1.014 m from the tank base or
= 3.2 – 1.014 = 2.186 m from the surface

You can use the method that you find easiest, both should give the same answer.

We are now required to find what head of water would give the same static pressure as the oil and water combined.

From F = ρghA, where A = b x d and h = $\frac{d}{2}$, so F = $\frac{\rho g d^2 b}{2}$

So 124.98 = 1000 x g x d^2 x $\frac{3}{2}$

So $d^2 = \frac{2 \times 124.98}{1000 \times g \times 3}$

Hence d = 2.914m

SAQ #4

A wood cube block floats freely in fresh water. When a 10kg steel mass is placed on top of the wood cube, 70% of the block is uniformly submerged.

a) Calculate the dimensions of the wood cube (6)

When an additional steel mass “m” is hung beneath the cube, the wooden block is just submerged.

b) Calculate the additional steel mass “m” (6)

a) Calculate the percentage of wood cube exposed if *both* steel masses are removed (4)

Take density of wood as 550 kg/m^3
Take density of steel as 7900 kg/m^3

ANS at end of the Hydrostatics unit

Student self-test

Complete these questions, and check your answers with those given at the end of each question.

SAQ

A spherical buoy 203 mm diam. just floats awash in sea water with a piece of steel of mass 3.175 kg suspended from its underside. If ρ for steel is 7800kg/m^3, find the mass of the buoy.

ANS 1.73 kg

SAQ

A block of wood is 1 m long and is of square cross section 250 mm by 250 mm. Its relative density is 0.9, and it floats in fresh water with its upper face horizontal. If one face of the wooden block is to be sheathed with metal of relative density 2.5, determine the mass of metal required to be attached,

(a) on the upper face,
(b) on the lower face,

so that in each case the upper face of the wooden block is at the water level.

ANS 10.4 kg, 6.3 kg

SAQ

A piece of metal is suspended from a spring balance. When the metal is in air, the balance reads 10 N. When the metal is completely submerged in fresh water, the spring balance reads 8.5 N, when the metal is completely submerged in oil, the spring balance reads 8.7 N. Calculate the density of the metal and the relative density of the oil.

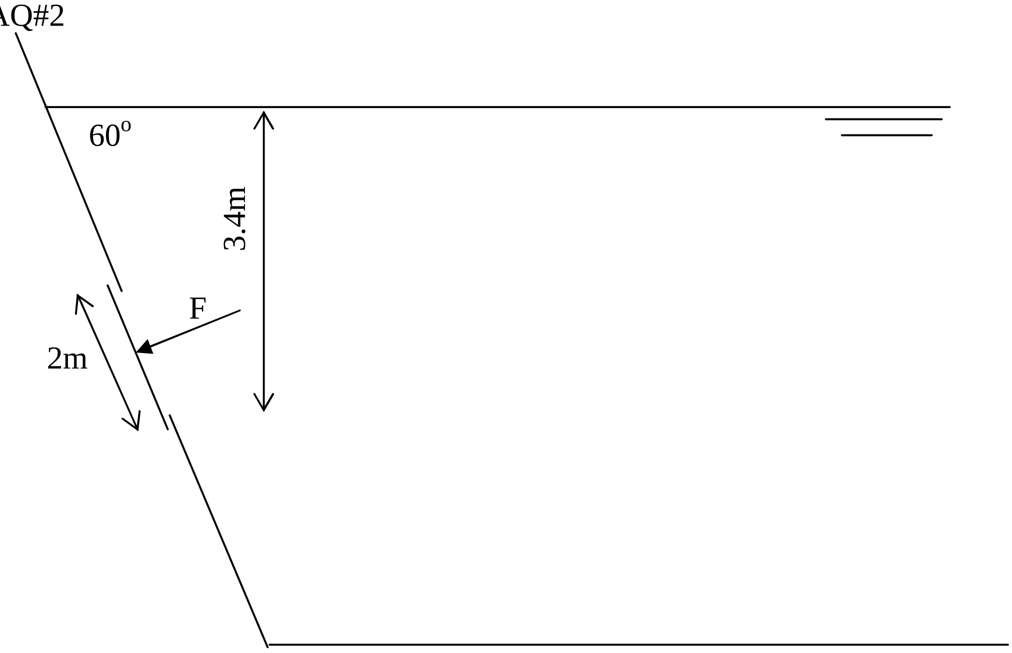
ANS 6670 kg/m^3, 0.866

YOU HAVE NOW COMPLETED THE HYDROSTATICS SECTION, AND SHOULD BE ABLE TO ATTEMPT THE MODULE ASSESSMENT QUESTIONS FOR THIS SECTION.

SAQ#1
Average depth of liquid = 0.7m.
Area of the tank side = 1.4 x 3 = 4.2 m^2

Force = ρghA = 1000 x g x 0.7 x 4.2 = 28.8kN

SAQ#2



STEP ONE - Obtain the wetted area of the door
Area of door = 2^2 = $4m^2$

STEP TWO – Obtain the h value and the static force
h is the vertical distance from the water level to the centre of area. For a square the centre of area is in the centre of the square or $\frac{d}{2}$,

so h = $3.4 - \frac{2\sin 60^o}{2}$ = 2.534m

Force = ρghA
Thus F = 1000 x g x 2.534 x 4 = 99.43kN

STEP THREE – Obtain the values of second moment of area, and then COP
This force will act at the centre of pressure, and the COP can be found in stages

I_{GG} for a square is $\frac{length^4}{12}$ = $\frac{2^4}{12}$ = $1.33m^4$

x **measured along the plane** = $\frac{3.4}{\sin\phi}$-1 = $\frac{3.4}{\sin 60^o} - 1$ = 2.925m

We have already calculated the centre of area depth and area, so the calculation of COP will be

y = $\frac{I_{GG}}{A \times x} + x$ = $\frac{1.33}{4 \times 2.925} + 2.925$ = 3.039m

SAQ#3

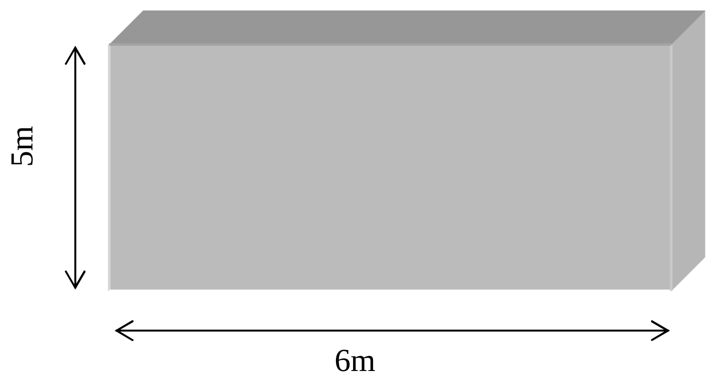
A tank of dimensions 6 m long, 3 m wide, and 5 metre deep is filled with fresh water. A pressure test is applied by filling the tank into the air vent pipe to give an additional head of 800mm. A 500mm circular access door is fitted with its centre 600mm from the tank floor. The access door is hinged at its top edge and secured by a single bolt at its lowest edge.

Calculate

a) The force on the access door

b) The tension on the bolt

c) The maximum permissible depth of water in the tank, if the bolt tension is limited to 4kN



STEP ONE - Obtain the wetted area of the door

Area of door = $\frac{\pi 0.5^2}{4}$ = 0.196m^2

STEP TWO – Obtain the h value and the static force

h is the vertical distance from the water level to the centre of area.

For this example the additional air vent static head must be considered so that the head on the door = h = 5.0 + 0.8 – 0.6 = 5.2m

Force = ρghA

Thus F = 1000 x g x 5.2 x 0.196 = 10kN

STEP THREE – Obtain the values of second moment of area, and then COP

This force will act at the centre of pressure, and the COP can be found in stages

I_{GG} for a circle is $\frac{\pi d^4}{64}$ = $\frac{\pi 0.5^4}{64}$ = 0.0031m^4

We have already calculated the centre of area depth and area, so the calculation of COP will be

$$y = \frac{I_{GG}}{A \times h} + h \quad = \quad \frac{0.0031}{0.196 \times 5.2} + 5.2 \quad = \quad 5.203\text{m}$$

A very minor difference between the COP and h values, but we should always calculate it to show our understanding of the background data required for this type of calculation.

STEP FOUR – Calculate, using moments, the force on the bolt
Evaluate moments about the hinge at the top edge
$(F \times 0.253) - (F_{Bolt} \times 0.5) = 0$
But F = 10kN, so $F_{Bolt} = 5.06$kN

STEP FIVE
This force is larger than the permissible force required in part c), where the bolt tension is limited to 4kN

If F_{Bolt} is limited to 4kN, then this can be achieved by reducing the water level, which will change the value of h. The values of area and I_{GG} will remain constant, thus restating the relationship for F

F = 1000 x g x h x 0.196 = 1922.76h

$$y = \frac{I_{GG}}{A \times h} + h \quad = \quad \frac{0.0031}{0.196 \times h} + h \quad = \quad \frac{0.0158}{h} + h$$

We have already calculated that y is 3mm below h, or 0.25 + y – h (study your diagram to see where this has been found)

Evaluating the moments about the top hinge,
gives $(F \times (0.25 + y - h)) - (F_{Bolt} \times 0.5) = 0$
or $(F \times (0.25 + 0.0158/h + h - h)) - (F_{Bolt} \times 0.5) = 0$
thus $(1922.76h \times (0.25 + 0.0158/h + h - h)) - (F_{Bolt} \times 0.5) = 0$

480.69h + 30.38 – (4000 x 0.5) = 0

$$\text{So } h = \frac{2000 - 30.38}{480.69} \quad = \quad 4.097\text{m}$$

Thus the water depth will be 0.6m greater than the h value, so water depth is 4.697m

SAQ #4

A wood cube block floats freely in fresh water. When a 10kg steel mass is placed on top of the wood cube, 70% of the block is uniformly submerged.

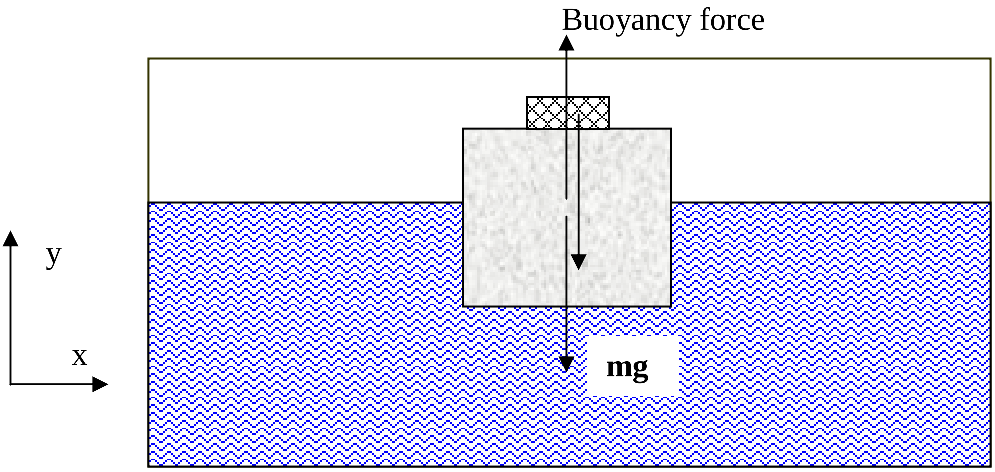
a) Calculate the dimensions of the wood cube (6)

When an additional steel mass "m" is hung beneath the cube, the wooden block is just submerged.

b) Calculate the additional steel mass "m" (6)

b) Calculate the percentage of wood cube exposed if *both* steel masses are removed (4)

Take density of wood as 550 kg/m^3
Take density of steel as 7900 kg/m^3



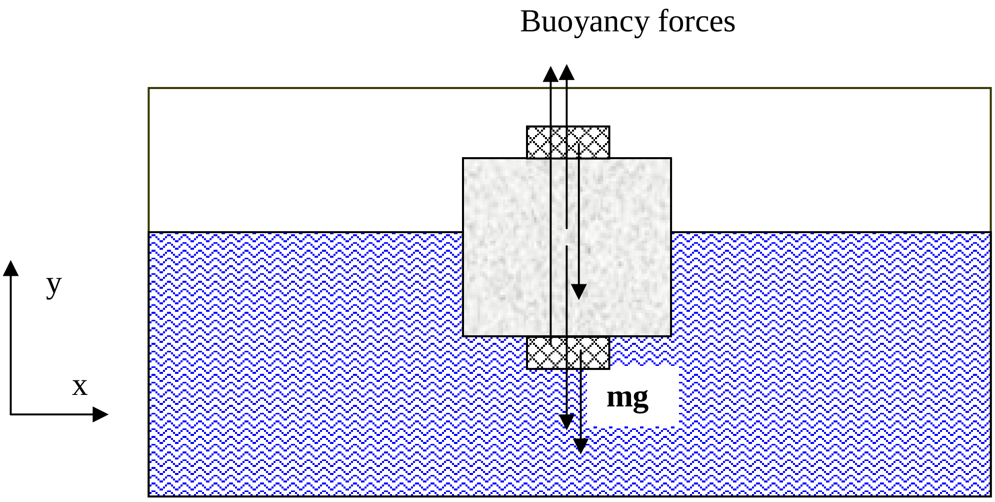
In this question, we have two masses that provide the gravitational force, the mass of the wood cube and the mass of the steel block. Equating the forces in the y direction:

$\rho Vg - m_s g - m_w g = 0$, rewriting the mass of the wood, m_w, as $\rho_w V_w$

$1000 \times 0.7\, L^3 \times g - 10g - 550 \times L^3 \times g = 0$, and divide all terms by g, gives

$(700 - 550)L^3 - 10 = 0$

So $L^3 = 10/150$, so L = 405.5mm



The addition of the steel mass beneath the wood cube will provide a gravitational force *and* buoyancy force as it is displacing water, so we will now have two buoyancy forces and three gravitational forces. Check that you can see these forces.

The volume of the wood cube immersed this time will be 100% as it is just immersed.
The volume of the steel mass fixed beneath can be equated to m/ρ_s

Equating the forces in the y direction:
$\rho V_w g + \rho V_s g - m_s g - m_s g - m_w g = 0$

$$1000 \text{ x } (0.4055)^3 \text{ x } g + 1000 \text{ x } \left(\frac{m_s}{7900}\right) \text{ x } g - m_s g - 10g - 550 \text{ x } (0.4055)^3 \text{ x } g = 0$$

$654.1 + 1.24m_s - 9.81m_s - 98.1 - 359.8 = 0$

$196.2 - 8.57m_s = 0$

So $m_s = 22.9\text{kg}$

In this final part we must now consider the wood cube floating in isolation, so only one buoyancy and one gravitational force will present. If we call the percentage of cube submerged as "y", then equating the forces in the y direction:

$\rho V_w g - m_w g = 0$

$1000 \text{ x } y \text{ x } (0.4055)^3 \text{ x } g - 550 \text{ x } (0.4055)^3 \text{ x } g = 0$

So $1000y - 550 = 0$, so $y = 0.55$

Thus there will be 100 – 55 or 45% of the cube exposed.