# TEMPERATURE STRESSES IN COMPOUND BARS

## Thermal Stress and Strain

If a metal component is heated it will expand. If it is allowed to expand freely without any restrictions then there can be NO stress. A stress is only caused when a force is applied to a component. However, because the length of the material changes, there WILL be a strain. This is called thermal strain and its value is given by

$$\text{Thermal Strain} = \frac{\Delta L}{L} \text{ as with any strain.}$$

$$\text{However, in this case } \Delta L = \alpha \times L \times \Delta t$$

$$\text{where } \alpha = \text{coeffiecient of linear expansion /°C}$$

$$\Delta t = \text{change in temperature °C}$$

α depends on the material.

If the component is RESTRICTED from expanding then a thermal stress will be set up due to the restricting force being applied. In this case the change in length ΔL will be prevented and the stress value can be found from

$$\Delta L = \frac{\sigma \times L}{E} = \alpha \times L \times \Delta t$$

If the change in length is only partly restricted, then the stress value will be less depending upon the value of ΔL.

**WORKED EXAMPLE**

A piece of steel is 1 m long and is heated from $18^0$C to $100^0$C. α for steel is $12 \times 10^{-6}$ /$^0$C.

(a) What is the thermal strain for unrestricted expansion?

$$\varepsilon = \frac{\alpha \times L \times \Delta t}{L} = \alpha \times \Delta t$$

$$\varepsilon = 12 \times 10^{-6} \times (100 - 18)$$

$$\varepsilon = \underline{\mathbf{0.000984}} \text{ Answer}$$

(b) What is the change in length for unrestricted expansion?

$$\Delta L = \varepsilon \times L$$

$$\Delta L = 0.000984 \times 1000 \text{ mm}$$

$$\Delta L = \underline{\mathbf{0.984\ mm}} \text{ Answer}$$

(c) If the change in length calculated in part (b) was totally restricted, what would be the thermal stress? Take E for steel as 207 GN/m$^2$.

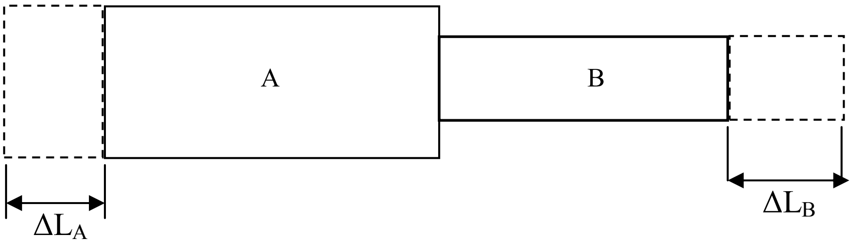
$$\sigma = \varepsilon \times E$$

$$\sigma = 0.000984 \times 207 \times 10^9$$

$$\sigma = \underline{\mathbf{203.7\ MN/m^2}} \text{ Answer}$$

Notice that even a small temperature change (82 $^0$C) on a short length of steel (1 m) produces a very large stress for total restriction (203.7 MN/m$^2$). The effects of temperature change should not be ignored and can produce very large stresses which may cause failure.

**Thermal Stress and Strain in Compound Bars in Series**

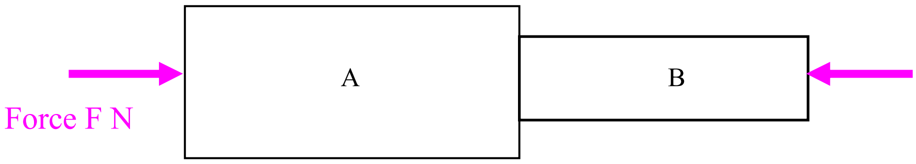



The two components A and B may be different materials (e.g. one steel and one copper), they may be different lengths, they may be different cross sectional areas. When heated through the same temperature change, each component will expand a different amount depending on the length and coefficient of linear expansion.

The total change in length of the compound bar will be

$$\Delta L_{TOTAL} = \Delta L_A + \Delta L_B$$
$$\Delta L_{TOTAL} = \alpha_A \times L_A \times \Delta t + \alpha_B \times L_B \times \Delta t$$

If the expansion is restricted, then a stress will be set up.




$$\text{The force F will be } F = \sigma_A \times A_A = \sigma_B \times A_B \quad \text{Equation (1)}$$

$$\text{Also } \Delta L_A = \frac{\sigma_A \times L_A}{E_A} \text{ and } \Delta L_B = \frac{\sigma_B \times L_B}{E_B}$$

$$\text{Therefore } \frac{\sigma_A \times L_A}{E_A} + \frac{\sigma_B \times L_B}{E_B} = \alpha_A \times L_A \times \Delta t + \alpha_B \times L_B \times \Delta t \quad \text{Equation (2)}$$

From the two equations it is possible to find the two stresses.

**WORKED EXAMPLE**

A steel bar is 300 mm long. It is 24 mm diameter for 200 mm of the length and 18 mm diameter for the remainder of the length. It is heated through a temperature rise of 30 $^0$C. Take E for steel as 207 GN/m$^2$ and α as 12 × 10$^{-6}$ /$^0$C

(a) Find the total extension of the bar for unrestricted expansion.

$$\Delta L_{TOTAL} = \Delta L_{24} + \Delta L_{18}$$
$$\Delta L_{TOTAL} = (\alpha \times L \times \Delta t)_{24} + (\alpha \times L \times \Delta t)_{18}$$
$$\Delta L_{TOTAL} = (12 \times 10^{-6} \times 200 \times 30) + (12 \times 10^{-6} \times 100 \times 30)$$
$$\Delta L_{TOTAL} = \underline{\mathbf{0.108\ mm}}\ \text{Answer}$$

(b) The steel is now clamped securely at each end and allowed to cool to its original temperature. Find the stresses in each section of the bar.

$$\Delta L_{TOTAL} = \frac{\sigma_A \times L_A}{E_A} + \frac{\sigma_B \times L_B}{E_B}$$
$$0.000108\ \text{m} = \frac{\sigma_A \times 0.2}{207 \times 10^9} + \frac{\sigma_B \times 0.1}{207 \times 10^9}$$

Multiply both sides by $207 \times 10^9$

$$22.36 \times 10^6 = 0.2 \times \sigma_A + 0.1 \times \sigma_B \quad \text{Equation (1)}$$

$$F = \sigma_A \times A_A = \sigma_B \times A_B$$
$$\sigma_A \times \frac{\pi}{4} \times 0.024^2 = \sigma_B \times \frac{\pi}{4} \times 0.018^2$$
$$\sigma_A = 0.5625 \times \sigma_B \qquad \text{Equation (2)}$$

Substitute (2) into (1)

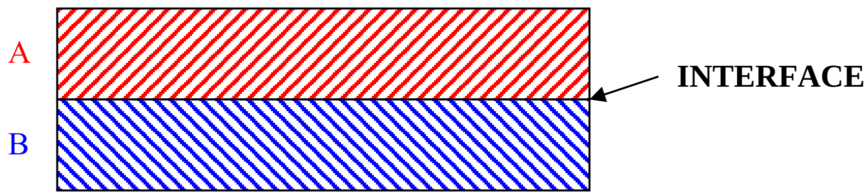
$$22.36 \times 10^6 = 0.2 \times 0.5625 \times \sigma_B + 0.1 \times \sigma_B$$
$$22.36 \times 10^6 = 0.2125 \times \sigma_B$$

Therefore $\sigma_B = \underline{\mathbf{105.2\ MN/m^2}}$ Answer

From Equation (2) $\sigma_A = \underline{\mathbf{59.19\ MN/m^2}}$ Answer

**Temperature Stress and Strain in Compound Bars in Parallel**

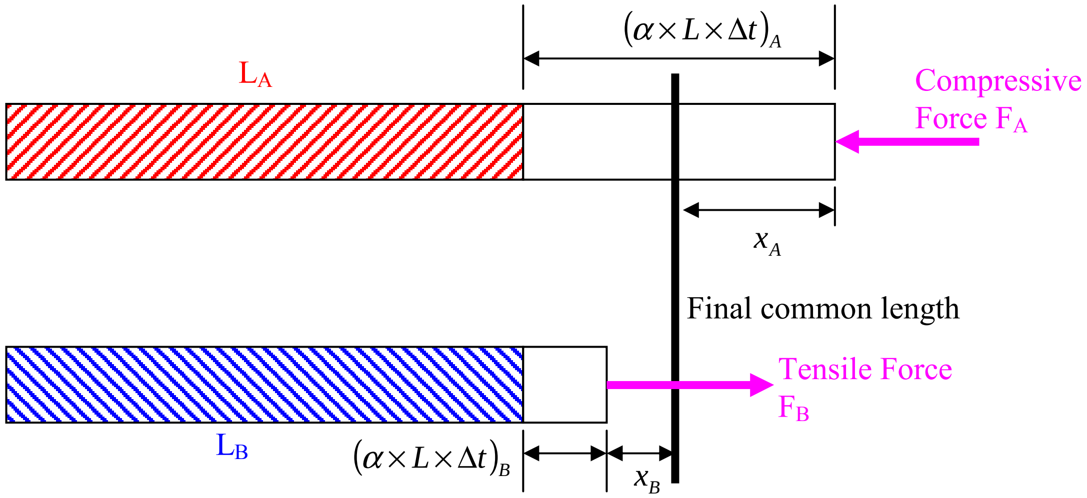
If the two components are parallel with each other then some assumptions are made:




1. The original lengths of each component are the same.
2. The final lengths of each component after expansion are the same.
3. If there is an INTERFACE (i.e. a plane where the materials are joined together), then no SLIP occurs between the materials at the interface.

For problems involving bolts through cover plates, shafts in bushes etc. the first two assumptions still apply.

When the component is heated, both pieces of material try to expand to their free lengths. However, due to no slip occurring at the interface, one material (the one with the largest coefficient of linear expansion) is prevented from expanding to its full free length. A compressive force is acting on it. The other piece of material (the one with the smaller coefficient of linear expansion) is pulled to a longer length than its free length. A tensile force is acting on it. Both pieces are assumed to end up at the same final length.




The assumption is that $\alpha_A$ is larger than $\alpha_B$.

From geometry, the changes in lengths to give the final common length are:

$$(\alpha_A \times L_A \times \Delta t) - x_A = (\alpha_B \times L_B \times \Delta t) + x_B$$

Therefore $x_B + x_A = (\alpha_A \times L_A \times \Delta t) - (\alpha_B \times L_B \times \Delta t)$

$$\frac{\sigma_A \times L_A}{E_A} + \frac{\sigma_B \times L_B}{E_B} = (\alpha_A \times L_A \times \Delta t) - (\alpha_B \times L_B \times \Delta t)$$

If the original lengths are the same then $L_A = L_B$

Therefore $\frac{\sigma_A}{E_A} + \frac{\sigma_B}{E_B} = (\alpha_A \times \Delta t) - (\alpha_B \times \Delta t)$

or $\underline{\frac{\sigma_A}{E_A} + \frac{\sigma_B}{E_B} = \Delta t(\alpha_A - \alpha_B)}$ Equation (1)

From equilibrium of forces

$$F_A - F_B = 0$$
$$F_A = F_B$$
$$\underline{\sigma_A \times A_A = \sigma_B \times A_B} \qquad \text{Equation (2)}$$

From these two equations, the two stresses can be calculated.

## WORKED EXAMPLE

A piece of Aluminium of cross sectional area 60 mm$^2$ is bonded to a piece of steel of cross sectional area 120 mm$^2$. The lengths of both materials are the same at 150 mm. The component is then heated through a temperature rise of 100 $^0$C.
Take $E_S$ = 207 GN/m$^2$ and $E_{AL}$ = 45 GN/m$^2$

$\alpha_S = 12 \times 10^{-6}$ /$^0$C and $\alpha_{AL} = 27 \times 10^{-6}$ /$^0$C.

(a) Find the stresses induced in the steel and aluminium.
(b) Find the extension of the compound bar.

(a)
From the equation for temperature stresses in a compound bar:

$$\frac{\sigma_{AL}}{E_{AL}} + \frac{\sigma_S}{E_S} = \Delta t(\alpha_{AL} - \alpha_S)$$

$$\frac{\sigma_{AL}}{45\times10^9} + \frac{\sigma_S}{207\times10^9} = 100\times\left(27\times10^{-6} - 12\times10^{-6}\right)$$

Multiply by $207\times10^9$

$$4.6\times\sigma_{AL} + \sigma_S = 310.5\times10^6 \quad \text{Equation (1)}$$

$$\sigma_{AL}\times A_{AL} = \sigma_S \times A_S$$

$$\sigma_{AL}\times120 = \sigma_S\times60$$

$$\sigma_{AL} = 2\times\sigma \qquad \text{Equation (2)}$$

Substitute (2) into (1)

$$4.6\times2\times\sigma_S + \sigma_S = 310.5\times10^6$$

$$10.2\times\sigma_S = 310.5\times10^6$$

$$\sigma_S = \underline{\mathbf{30.44\ MN/m^2}} \text{ Answer}$$

$$\sigma_{AL} = 2\times\sigma_S$$

$$\sigma_{AL} = \underline{\mathbf{60.88\ MN/m^2}} \text{ Answer}$$

(b)

The overall change in length of the compound bar is either given by:

$$\Delta L_{AL} = (\alpha \times L \times \Delta t)_{AL} - \left(\frac{\sigma \times L}{E}\right)_{AL}$$

$$\Delta L_{AL} = (27 \times 10^{-6} \times 150 \times 100) - \left(\frac{60.88 \times 10^{6} \times 150}{45 \times 10^{9}}\right)$$

$$\Delta L_{AL} = 0.405 - 0.2029$$

$$\Delta L_{AL} = \underline{\mathbf{0.2021\ mm}} \text{ Answer}$$

or

$$\Delta L_{S} = (\alpha \times L \times \Delta t)_{S} + \left(\frac{\sigma \times L}{E}\right)_{S}$$

$$\Delta L_{S} = (12 \times 10^{-6} \times 150 \times 100) + \left(\frac{30.44 \times 10^{6} \times 150}{207 \times 10^{9}}\right)$$

$$\Delta L_{S} = 0.18 + 0.02206$$

$$\Delta L_{S} = \underline{\mathbf{0.2021\ mm}} \text{ Answer}$$

i.e. the answer is the same whichever method is used.

Typical examination question

A steel rod of diameter 24mm is threaded at both ends and located within a aluminium tube 400 mm long with an internal diameter of 30mm and an external diameter of 55mm. The rod and tube assembly are fitted with end plates, and locked by a nut at each end. Each nut is tightened half a turn, on a thread of pitch 0.8mm.

Calculate EACH of the following

a) The stress in both the aluminium tube and steel rod following tightening (9)

b) The maximum permissible change in temperature allowed if the rod stress must not exceed 280 $MN/m^2$ (4)

c) The change in temperature required following tightening, for the rod to become slack within the aluminium tube (3)

Take the Modulus of Elasticity for steel as 200 $GN/m^2$

Take the Modulus of Elasticity for aluminium as 80 $GN/m^2$

Take the coefficient of thermal expansion for steel as $12 \times 10^{-6}/^{o}C$

Take the coefficient of thermal expansion for aluminium as $22 \times 10^{-6}/^{o}C$

**This question is a compound bar arranged in parallel. Hence the force within the steel rod must equal the force within the aluminium tube.**

We are also informed that the rod is extended by the tightening of the nut at each end. Thus the total extension is (2 x 0.5 turn) x 0.8mm pitch = 0.8mm

This total strain will be split between the steel and the aluminium. Why? Well when the nut is tightened two things will occur, the steel bar will extend AND the aluminium tube will compress. If the steel was infinitely stiff, then the movement of the nut would compress the tube only. We have an unknown extension of the rod and compression of the tube, but we do know that the summation is 0.8 mm.

So as we know the force in each material has to be the same, then

$$F_S = F_A = \frac{E_A A_A x_A}{l_A} = \frac{E_S A_S x_S}{l_S}$$

$$A_A = \frac{\pi\left(0.055^2 - 0.03^2\right)}{4} = 1.67 \times 10^{-3}\ m^2$$

$$A_S = \frac{\pi\left(0.024^2\right)}{4} = 0.452 \times 10^{-3}\ m^2$$

From $\frac{E_A A_A x_A}{l_A} = \frac{E_S A_S x_S}{l_S}$, then $\frac{80 \times 1.67 \times x_A}{0.4} = \frac{200 \times 0.452 \times x_S}{0.4}$

Thus $x_A = 0.677\ x_S$

As $x_{total}$ = 0.8mm = $x_A + x_S$, then $0.8 = x_S + 0.677\ x_S$

So $x_S$ = 0.477 mm, and $x_A$ = 0.323 mm

From $\sigma = \frac{Ex}{l}$ then $\sigma_S = \frac{200\times10^9\times0.477\times10^{-3}}{0.4} = 238.5$ MN/m$^2$ (TENSILE)

and $\sigma_A = \frac{80\times10^9\times0.323\times10^{-3}}{0.4} = 64.6$ MN/m$^2$ (COMPRESSIVE)

NOTE the importance of recognising that the steel is in tension and the tube in compression. This statement is vital to include even if not specifically asked for within the question.

When the compound bar is heated, then the aluminium will expand at a higher rate than the steel bar, thus causing a tensile force in the steel bar in addition to the existing compressive stress. We have been informed in the question stem that this increase to limited to a total maximum of 280 MN/m$^2$

The maximum allowable increase in the tensile stress is 280 – 238.5 = 41.5 MN/m$^2$

From the thermal strain relationship of $\frac{\sigma_A}{E_A} + \frac{\sigma_S}{E_S} = (\alpha_A - \alpha_S)\Delta t$

We know that the relationship between the stress in the steel and stress in the aluminium is related, as the force in each must equate, so $\sigma_A A_A = \sigma_S A_S$, thus

$\sigma_A = \sigma_S \frac{A_S}{A_A} = \sigma_S \frac{0.452}{1.67} = 0.27\sigma_S$

Thus $\frac{0.27\sigma_s}{80\times10^9} + \frac{\sigma_S}{200\times10^9} = (22-12)\times10^{-6}\Delta t$, and substituting for the known stress, $\sigma_S$ of 41.5 MN/m$^2$, gives a temperature increase of 34.8 $^oC$

We can also use the same relationship found above for the final part. This time the temperature must fall for the tension in the bar to be reduced. Thus $\sigma_S$ must now fall by 238.5 MN/m$^2$, so inserting this value into the equation gives

$$\frac{0.27\times238.5\times10^6}{80\times10^9} + \frac{238.5\times10^6}{200\times10^9} = (22-12)\times10^{-6}\Delta t$$

Thus $\Delta t = 200^oC$

## STUDENT EXAMPLES

**1.** A compound bar is made up of a central steel plate of 50 mm × 20 mm section bounded by two copper plates, each of 50 mm × 10 mm section, rigidly fastened to either side. At normal temperature the length of the assembly is 1.25 m. If the temperature is raised by 100 $^0$C find the stress in each material and the extension of the bar.
$E_{steel}$ = 200 GN/m$^2$; $E_{copper}$ = 100 GN/m$^2$; $\alpha_{steel}$ = 12 × 10$^{-6}$ /$^0$C; $\alpha_{copper}$ = 18 × 10$^{-6}$ /$^0$C.

($\sigma_{steel}$ = 40 MN/m$^2$; $\sigma_{copper}$ = 40 MN/m$^2$; 1.75 mm)

**2.** The cross sectional area of the flange of a cast iron cylinder cover is ten times the total cross sectional area of the steel studs securing the cover to the cylinder. If the cover joint starts to blow when the temperature is raised by 200 $^0$C, what was the initial stress in the studs.
$E_{steel}$ = 200 GN/m$^2$; $E_{ci}$ = 100 GN/m$^2$; $\alpha_{steel}$ = 12 × 10$^{-6}$ /$^0$C; $\alpha_{ci}$ = 11 × 10$^{-6}$ /$^0$C.

(33.33 MN/m$^2$ tensile)

**3.** Two copper pipes of 100 mm outside diameter are joined by brass flanges of 200 mm outside diameter. The flanges are fastened together by six 20 mm diameter steel bolts and when cold the nuts are just tight. Assuming The stress in the flange is uniformly distributed find the stress in the bolts when the temperature is raised by 120 $^0$C.
$E_{steel}$ = 200 GN/m$^2$; $E_{brass}$ = 80 GN/m$^2$; $\alpha_{steel}$ = 12 × 10$^{-6}$ /$^0$C; $\alpha_{brass}$ = 19 × 10$^{-6}$ /$^0$C.

(138 MN/m$^2$ tensile)

**4.** the length of a rod is eight times its diameter and when it is heated through 70 $^0$C its expansion is restricted to 0.25 mm. If the compressive stress due to thermal change is not to exceed 60 MN/m$^2$ determine the required dimensions of the rod.
E = 210 GN/m$^2$; $\alpha$ = 12 × 10$^{-6}$ /$^0$C.

(length 450 mm; diameter 56.25 mm)

**5.** A compound bar is made up of a steel bar of cross sectional area 40 cm$^2$ and a copper bar of cross sectional area 30 cm$^2$ each of the same length. If the temperature of the assembly is now raised by 250 $^0$C, find the stresses in each material and the linear coefficient for the compound bar.
$E_{steel}$ = 200 GN/m$^2$; $E_{copper}$ = 100 GN/m$^2$; $\alpha_{steel}$ = 12 × 10$^{-6}$ /$^0$C; $\alpha_{copper}$ = 18 × 10$^{-6}$ /$^0$C.

($\sigma_{steel}$ = 81.8 MN/m$^2$ tensile; $\sigma_{copper}$ = 40 MN/m$^2$ compressive; 13.64 × 10$^{-6}$ /$^0$C)

**6.** In a refrigeration plant a copper bolt of 12 mm diameter is fitted with a steel distance piece of 15 mm bore and a wall thickness of 3 mm. When fitted at 15 $^0$C the stress in the bolt was 9 MN/m$^2$. If when the plant is in operation the temperature falls to –25 $^0$C, find the resultant stresses in the bolt and the distance piece.
$E_{steel}$ = 200 GN/m$^2$; $E_{copper}$ = 100 GN/m$^2$; $\alpha_{steel}$ = 12 × 10$^{-6}$ /$^0$C; $\alpha_{copper}$ = 18 × 10$^{-6}$ /$^0$C.

($\sigma_{steel}$ = 18 MN/m$^2$ compressive; $\sigma_{copper}$ = 27 MN/m$^2$ tensile)

**7.** A steel tie rod of 25 mm diameter is placed concentrically in a brass tube of 60 mm bore and 70 mm outside diameter. They are fastened together by means of nuts and washers to form a rigid assembly with the nuts initially tightened to give a compressive stress of 40 MN/$m^2$ in the tube, after which a tensile force of 20 kN is applied to the tie rod. If the rod and tube have the same effective length, find the resultant stresses in them
(a) When there is no change of temperature.
(b) When the temperature increases by 40 $^0$C.
$E_{steel}$ = 210 GN/$m^2$; $E_{brass}$ = 90 GN/$m^2$; $\alpha_{steel} = 12 \times 10^{-6}$ /$^0$C; $\alpha_{brass} = 18.5 \times 10^{-6}$ /$^0$C.

((a) $\sigma_{steel}$ = 124.2 MN/$m^2$ tensile; $\sigma_{brass}$ = 20 MN/$m^2$ compressive
(b) $\sigma_{steel}$ = 149.9 MN/$m^2$ tensile; $\sigma_{brass}$ = 32.38 MN/$m^2$ compressive)

**8.** Three bars each of 12 mm × 6 mm section are riveted together to form a composite bar of 12 mm × 18 mm section. The outer bars are of aluminium and the centre bar is steel.
(a) If an axial compressive force of 10 kN is applied to the compound bar, find the stress in each material and the reduction in length on a gauge length of 200 mm.
(b) If the temperature is now raised by 100 $^0$C, determine the final stresses.
$E_{steel}$ = 210 GN/$m^2$; $E_{alum}$ = 70 GN/$m^2$; $\alpha_{steel} = 12 \times 10^{-6}$ /$^0$C; $\alpha_{alum} = 25 \times 10^{-6}$ /$^0$C.

((a) $\sigma_{steel}$ = 83.4 MN/$m^2$ compressive; $\sigma_{alum}$ = 27.8 MN/$m^2$ compressive; 0.0795 mm
(b) $\sigma_{steel}$ = 25.8 MN/$m^2$ tensile; $\sigma_{alum}$ = 82.2 MN/$m^2$ compressive)

**9.** Making use of the information given in the previous question, determine the increase in temperature required, after application of the compressive force of 10 kN, so that the final stress in each material will be the same. Find also the linear coefficient for the composite bar.

(33 $^0$C; $17.2 \times 10^{-6}$ /$^0$C)